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An asymptotic, large time solution of the convection Stefan problem with surface radiation

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Abstract—An asymptotic, large time solution has been obtained for the convection Stefan problem with surface radiation. The moving boundary problem has been reformulated as a fixed boundary problem where Lagrange–Bürmann expansions are used to complete the variable transformation. An asymptotic solution of the problem is obtained by requiring that the asymptotic expansions assumed for the interface position $X(t)$ and wall temperature $u_w(t)$ for large times are consistent with the resulting interfacial Lagrange–Bürmann expansions. It is found that the asymptotic expansions admit Neumann's solution as the leading terms and that logarithmic terms start intervening at the third-order terms of the expansions for nonzero Stefan number.

1. INTRODUCTION

THIS PAPER presents an asymptotic, large time solution of the convection Stefan problem with surface radiation. Reviewing the literature on the Stefan problems, we find very few works on asymptotic solutions particularly at large times. Leading terms of the large time expansions for one-phase Stefan problems seem to have first been obtained by Cannon and Denson Hill [1] for temperature boundary specification and by Cannon and Primicerio [2] for flux boundary specification. Using Stokes' theorem, they formulate the Stefan problem of what Katz [3] classifies as finite bar configuration (see the remarks just below) in such a way as to allow an asymptotic evaluation of the leading term of the interface position $X(t)$ as $t \to \infty$. A more general approach is developed by Katz [3] for obtaining large time solutions of Stefan problems whereby asymptotic expansions are evaluated directly from an integro-differential representation of the Stefan problem, the approach originally due to Portnov [4] in the classical series expansion method. He also obtains leading terms of the asymptotic expansions for two classes of problems, one for the semi-infinite, the other for the finite bar configurations. From the analytical point of view, the latter is the more involved because of finiteness in dimension. For example, a large time solution of a crystal growth Stefan problem given in ref. [5] belongs to the semi-infinite bar configuration in Katz's classification because one is only concerned with the concentration field in the melt system which extends over a semi-infinite domain. In spite of added complexities at the interfacial conditions due to the limiting rate of crystal growth, the asymptotic analysis was carried out without difficulty in inverse powers of time as in many classical asymptotic expansions for large values of coordinates (see ref. [6] for various examples). Some special technique must be devised to deal with the present case of finite bar. Our key strategy here is to reformulate the moving boundary problem as a fixed boundary problem.

In this paper, we want to present an asymptotic solution, valid as $t \to \infty$, of the one-phase convection Stefan problem with Newton's radiation condition. After formulating the problem in Section 2, the moving boundary problem is reformulated as a fixed boundary problem whereby the method of Lagrange–Bürmann (LB) expansions plays a key role in completing the variable transformation. Here the new time-like variable τ must be introduced such that $\tau = O(t^{-1/2})$ as $t \to \infty$. In the process, we find the nonlinear interfacial LB expansions for $Xu_w/(1-u_w)$ and $X\dot{X}/(1-u_w)$ must balance the heat flux values at both ends of the growing phase, the latter depending on the temperature field. In the large time limit of $t \to \infty$, neither the temperature function nor the mapping functions nor the LB expansions are expected to be analytic at $\tau = 0$ or as $t \to \infty$ because neither X nor u_w are expected to be so in these regions. Therefore, we have to construct consistent asymptotic expansions for the temperature field, the LB expansions and the X and u_w functions. The temperature functions are obtained in Section 4. The validity of the asymptotic expansions will be demonstrated in Section 5 by showing that the coefficients in the interfacial LB expansions as well as in X and u_w functions can be determined with consistency. An inversion of the LB expansions is presented in Section 6. We find that the leading terms of the expansions for X and u_w admit the Neumann solution and that logarithmic terms start intervening at the third-order term of $O(t^{-1/2})$ and $O(t^{-3/2})$, respectively. Within the framework of the present analysis, there is one constant C_{31} which may not be determined, reflecting the fact that the initial condition may not be enforced. In Section 7, we have determined the constant by patching with the numerical solution of the problem. We then get a good agreement down to as low as $X = 2$. For a mathematical justification for the construction of LB expansions from formal (merely asymptotic) series

NOMENCLATURE

$\{a_i\}, \{b_i\}$	coefficients of Lagrange–Bürmann expansions for $\dot{u}_w X^2/(1-u_w)$ and $\bar{h}X^2$
$\{C_i\}, \{d_i\}$	coefficients of asymptotic expansions for $X(t)$ and $u_w(t)$
$g_1(t), g_2(t)$	mapping functions defined by $Xu_w/(1-u_w)$ and $X\dot{X}/(1-u_w)$, respectively
$\bar{h}(t)$	mapping function defined by $X\dot{X}-2$
$p_1^*(\tau), p_2^*(\tau)$	composite functions given by $g_1\{h^{-1}\}$ and $g_2\{h^{-1}\}$
t	dimensionless time
$\bar{t}$	$\alpha^2 t$
$u(\eta, \tau)$	dimensionless temperature
$u_w(t)$	dimensionless wall temperature
$\{u_i(\eta)\}$	series solution of $u(\eta, \tau)$
x	dimensionless distance
X	dimensionless interface position
Y	$\sqrt{\varepsilon}\cdot\alpha\eta$.

Greek symbols

α	Neumann constant
ε	Stefan number
η	fixed boundary coordinate defined as x/X
τ	time-like variable defined by $h(t)$.

Superscripts and other symbols

(k)	kth approximations based on the truncated LB expansions
$\dot{}$	derivative with respect to time
$'$	derivative with respect to η
-1	inverse function.

such as the present ones, the reader is referred to Chap. 1.9 of Henrici [7, p. 55].

2. FORMULATION

Let the heating or cooling surface which effectively initiates the change of phase in this problem satisfy Newton's law of radiation. The one-dimensional, one-phase Stefan problem can be formulated as

$$\varepsilon\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial t^2}, \tag{1}$$

$$\frac{\partial \bar{u}}{\partial x}(0,t) = \bar{u}(0,t) = u_w(t), \tag{2}$$

$$\frac{\partial \bar{u}}{\partial x}(X,t) = \frac{\mathrm{d}X}{\mathrm{d}t}, \tag{3}$$

$$\bar{u}(X,t) = 1, \tag{4}$$

$$X(0) = 0. \tag{5}$$

The Stefan number ε and the dimensionless quantities x, X, ε, t, $\bar{u}$ and u_w are related to the dimensional quantities x^*, X^*, t^*, u^* and u_w^* as

$$\varepsilon = \frac{c(u_s-u_0)}{L}, \quad x = \frac{\gamma}{k}x^*, \quad X = \frac{\gamma}{k}X^*,$$
$$t = \frac{\varepsilon\gamma^2}{\rho k c}t^*, \quad \bar{u} = \frac{u^*-u_0}{u_s-u_0}, \quad u_w = \frac{u_w^*-u_0}{u_s-u_0} \tag{6}$$

where c is the specific heat, ρ the density, L the latent heat, k the thermal conductivity, u_s, u_0, u_w^* are the phase-change temperature, the coolant temperature and the wall temperature, respectively and γ is the heat transfer coefficient by Newton's law of radiation.

In the limit of $t \to \infty$, we physically expect the solution of the problem to approach Neumann's solution because u_w^* above settles down, at such a thermodynamic equilibrium state, to the coolant temperature u_0 so that the boundary conditions reduce to that of temperature specification. Neumann's exact solution is given as (see [8] for example)

$$X \sim 2\alpha\sqrt{t}, \tag{7}$$

$$\alpha\,\mathrm{erf}(\sqrt{\varepsilon}\cdot\alpha)\exp(\varepsilon\alpha^2) = \sqrt{\varepsilon/\pi}. \tag{8}$$

Here α is Neumann's constant satisfying the transcendental equation of (8). In series form, α is given as

$$\alpha = \frac{1}{\sqrt{2}}\left(1 - \frac{\varepsilon}{6} + \frac{31}{720}\varepsilon^2 + \ldots\right). \tag{9}$$

Evidence is available in the literature to verify this asymptotic behavior. The analysis in [2] shows that

$$X(t) \sim \beta\int_0^t u_w(\zeta)\,\mathrm{d}\zeta \quad \text{as } t \to \infty \tag{10}$$

where β is some constant less than unity satisfying, as $t \to \infty$

$$1\Big/\left\{1 + \frac{1}{\sqrt{\pi}}\int_0^t \frac{u_w(\zeta)\,\mathrm{d}\zeta}{\sqrt{t-\zeta}}\,\mathrm{d}\zeta\right\} \leqslant \beta \leqslant 1.$$

Since our analysis establishes that $u_w(t) \sim \alpha\exp(\varepsilon\alpha^2)/\sqrt{t}$ [see equation (61)], $X \sim 2\alpha_0\sqrt{t}$ follows although the constant α_0 must be determined by a more rigorous approach such as ours. Other evidence is found in the case of $\varepsilon = 0$. An exact solution for this case is given in [9] as,

$$X = \sqrt{1+2t} - 1, \tag{11}$$

$$u_w = \frac{1}{\sqrt{1+2t}}, \tag{12}$$

$$\bar{u} = \frac{1-x}{1-X}. \tag{13}$$

Examining the behavior of X and u_w as $t \to \infty$, we have

$$X \sim 2\sqrt{t/2} - 1 + \frac{1}{4\sqrt{t/2}} + \ldots,$$

$$u_w \sim \frac{1}{2\sqrt{t/2}} - \frac{1}{16(t/2)^{3/2}} + \ldots. \tag{14}$$

In the limit of $\varepsilon = 0$, (9) gives $\alpha = 1/\sqrt{2}$ thus recovering the Neumann solution of (7) as the leading term of (14). We want to establish this fact by demonstrating that the present problem indeed admits of such an asymptotic expansion, reproducing the asymptotic expansions of (14) for $\varepsilon = 0$ in particular.

3. FIXED BOUNDARY COORDINATES AND LB EXPANSIONS

Introducing the transformation of (15) below, the moving boundary will be transformed into a fixed boundary problem. The new time-like variable τ must be introduced such that τ is almost unity as $t \to \infty$.

$$\eta = \frac{x}{X}, \quad \text{and} \quad \tau = h(\bar{t}) = X\frac{\mathrm{d}X}{\mathrm{d}\bar{t}} - 2, \tag{15}$$

$$\bar{t} = \alpha^2 t. \tag{16}$$

The new time scale of (16) is introduced here merely to simplify the later notations. We show that α reduces to the Neumann constant satisfying (8). Assume that asymptotic expansions for X, u_w can be written as

$$X \sim 2\sqrt{\bar{t}} + C_2 + C_{30}\frac{\ln \bar{t}}{\sqrt{\bar{t}}} + C_{31}\frac{1}{\sqrt{\bar{t}}} + \ldots \tag{17}$$

$$u_w \sim \frac{d_1}{\sqrt{\bar{t}}} + \frac{d_2}{\bar{t}} + d_{30}\frac{\ln \bar{t}}{\bar{t}^{3/2}} + d_{31}\frac{1}{\bar{t}^{3/2}} + \ldots. \tag{18}$$

Hence we have

$$\tau = h(\bar{t}) \sim \frac{C_2}{\sqrt{\bar{t}}} + \frac{2C_{30}}{\bar{t}} + \ldots \tag{19}$$

$$\frac{1}{\sqrt{\bar{t}}} = h^{-1}(\tau) \sim \frac{\tau}{C_2} - \frac{2C_{30}}{C_2^3}\tau^2 + \ldots. \tag{20}$$

That logarithmic terms must intervene at $O(\bar{t}^{-1/2})$ and at $O(\bar{t}^{-3/2})$ for X and u_w becomes evident if the asymptotic expansions developed are consistent. It is interesting to note that logarithmic terms in X do not appear in the asymptotic expansions for the $h(\bar{t})$ function. The asymptotic expansions of (17) and (18) form the basic framework of the present analysis; it is essential that the temperature solution and the LB expansions are consistent with those forms assumed. Using the variables of (15), the governing equations and boundary conditions reduce to,

$$\frac{\partial^2 u}{\partial \eta^2} + \varepsilon\alpha^2(\tau+2)\eta\frac{\partial u}{\partial \eta} = \varepsilon\alpha^2 \underline{\frac{\dot{u}_w}{1-u_w}} X^2(1-u) + \varepsilon\alpha^2 \underline{\dot{h}X^2}\frac{\partial u}{\partial \tau}, \tag{21}$$

$$g_1(\bar{t}) = X\frac{u_w}{1-u_w} = g_1\{h^{-1}(\tau)\} = p_1(\tau) = \frac{\partial u}{\partial \eta}(0,\tau), \tag{22}$$

$$g_2(\bar{t}) = \frac{X\dot{X}\alpha^2}{1-u_w} = g_2\{h^{-1}(\tau)\} = p_2(\tau) = \frac{\partial u}{\partial \eta}(1,\tau), \tag{23}$$

$$u(0,\tau) = 0, \tag{24}$$

$$u(1,\tau) = 1, \tag{25}$$

$$X(0) = 0. \tag{26}$$

Here the dot denotes differentiation with respect to $\bar{t}$. u is redefined as

$$u = \frac{\bar{u} - u_w}{1-u_w}.$$

Since the analysis is valid only as $\bar{t} \to \infty$, the initial condition of (26) may not be enforceable. The two interfacial and boundary conditions of (2) and (3) now reduce to the LB expansions (22) and (23) for g_1 and g_2 functions, the RHS being equal to the heat fluxes to be specified at both ends of the interface. To construct the LB expansions, it is sufficient to develop asymptotic (formal) series expansions for g_1 and g_2 functions from (17) and (18) and then to compose with $h^{-1}(\tau)$ of (20) (see [7] for detailed discussions). This is followed below. From (17) and (18), we have

$$\begin{aligned} g_1 = X\frac{u_w}{1-u_w} &\sim 2d_1 + \frac{C_2 d_1 + 2(d_2 + d_1^2)}{\sqrt{\bar{t}}} + (C_{30}d_1 + 2d_{30})\frac{\ln \bar{t}}{\bar{t}} \\ &+ (2d_{31} + 2d_1^3 + 4d_1 d_2 + C_2 d_1^2 + C_2 d_2 + C_{31}d_1)\frac{1}{\bar{t}} + \ldots. \end{aligned} \tag{27}$$

By substituting (20) into $\bar{t}$ of (27), the LB expansion for $p_1(\tau)$ is now found to be

$$\begin{aligned} g_1\{h^{-1}(\tau)\} = p_1(\tau) &\sim 2d_1 + \frac{C_2 d_1 + 2(d_2 + d_1^2)}{C_2}\tau + \frac{2(d_1 C_{30} + 2d_{30})}{C_2^2}\tau^2 \ln\tau \\ &+ \left[\frac{2d_{31} + 2d_1^3 + C_{31}d_1 + 4d_1 d_2 + C_2 d_2 + C_2 d_1^2}{C_2^2} - \frac{2C_{30}(C_2 d_1 + 2d_2 + 2d_1^2)}{C_2^3} - \ln C_2^2\left(\frac{C_{30}d_1 + 2d_{30}}{C_2^2}\right)\right]\tau^2 + \ldots. \end{aligned} \tag{28}$$

In the same way, $p_2(\tau)$ is

$$g_2\{h^{-1}(\tau)\} = p_2(\tau) \sim 2\alpha^2 + \left(1+\frac{2d_1}{C_2}\right)\alpha^2\tau$$
$$+\frac{\alpha^2}{C_2^3}[2C_2(d_2+d_1^2)+C_2^2d_1-4d_1C_{30}]\tau^2+\ldots. \quad (29)$$

We require $p_1(\tau)$ and $p_2(\tau)$ to match with $\partial u(0,\tau)/\partial\eta$ and $\partial u(1,\tau)/\partial\eta$ term by term. Now to solve the temperature function of (21), the underlined terms must be expressed in terms of τ. This can be done as we have derived the p_1 and p_2 functions just above.

$$\dot{h}X^2 \sim -\frac{2C_2}{\sqrt{t}} - \frac{2(C_2+4C_{30})}{t} + \ldots = a_1\tau + a_2\tau^2 + \ldots \quad (30)$$

$$\frac{\dot{u}_w}{1-u_w}X^2 \sim -\frac{2d_1}{\sqrt{t}} - \frac{2(d_2+C_2d_1+d_1^2)}{t}$$
$$+\ldots = b_1\tau + b_2\tau^2 + \ldots \quad (31)$$

where

$$a_1 = -2, \quad a_2 = -2\left(1+\frac{2C_{30}}{C_2^2}\right)\ldots,$$
$$b_1 = -\frac{2d_1}{C_2},$$
$$b_2 = 2\left(\frac{2d_1C_{30}}{C_2^3} - \frac{d_2}{C_2^2} - \frac{d_1}{C_2} - \frac{d_1^2}{C_2^2}\right)\ldots. \quad (32)$$

Now substituting (30) and (31) into (21), we see that the temperature function $u(\eta,\tau)$ is a function of η and τ. Let us assume that u has the following asymptotic expansion for small τ or large t

$$u(\eta,\tau) \sim u_0(\eta) + \tau u_1(\eta) + \tau^2 u_2(\eta) + \ldots. \quad (33)$$

We will establish the validity of the asymptotic expansions (33), (17) and (18) by matching (28) and (29), term by term, with the RHS of (22) and (23) which inevitably depends on (33). We formulate $\{u_i\}$ functions in Section 4 and demonstrate the matching in Section 5.

4. TEMPERATURE FUNCTIONS

Now the temperature solution can be determined by substituting (30) and (31) into (21) subject to the boundary condition of (24) and (25). The appropriate governing equations and boundary conditions are then,

$$u_0'' + 2\varepsilon\alpha^2\eta u_0' = 0, \quad u_0(0) = 0, \quad u_0(1) = 1, \quad (34)$$
$$u_1'' + 2\varepsilon\alpha^2\eta u_1' + 2\varepsilon\alpha^2 u_1 = \varepsilon\alpha^2[b_1(1-u_0) - \eta u_0'], \quad (35)$$
$$u_2'' + 2\varepsilon\alpha^2\eta u_2 + 4\varepsilon\alpha^2 u_2 =$$
$$\varepsilon\alpha^2[a_2u_1 + b_2(1-u_0) - b_1u_1 - \eta u_1']\ldots, \quad (36)$$
$$u_i(0) = u_i(1) = 0 \quad \text{for} \quad i \geq 1.$$

In terms of $\{u_i\}$, the LB expansions of (22) and (23) reduce to

$$g_1 = X\frac{u_w}{1-u_w} = p_1(\tau) \sim u_0'(0)$$
$$+\tau u_1'(0) + \tau^2 u_2'(0) + \ldots, \quad (37)$$

$$g_2 = \frac{X\dot{X}\alpha^2}{1-u_w} = p_2(\tau) \sim u_0'(1)$$
$$+\tau u_1'(1) + \tau^2 u_2'(1) + \ldots. \quad (38)$$

Here the primes denote differentiation with respect to η. $\{u_i\}$ of (34), (35), etc. satisfy linear differential equations of error integral type and can be determined in closed form. Since undetermined constants a_1, a_2, etc. are involved in those equations, the best procedure is to execute the computations step by step, determining the coefficients of the LB expansions such that the expansions (28) and (29) are consistent with the asymptotic expansions of (37) and (38), the latter resulting from the solutions of $\{u_i\}$ of (34), (35), etc.

5. DETERMINING THE COEFFICIENTS OF THE LB EXPANSIONS

u_0 *and coefficient* d_1

u_0 satisfying (34) is given by

$$u_0(\eta) = \frac{\mathrm{erf}(\sqrt{\varepsilon}\cdot\alpha\eta)}{\mathrm{erf}(\sqrt{\varepsilon}\cdot\alpha)}. \quad (39)$$

We put $Y = \sqrt{\varepsilon}\cdot\alpha\eta$ hereinafter. Hence we have

$$u_0'(0) = \frac{2\sqrt{\varepsilon}\cdot\alpha}{\sqrt{\pi}\,\mathrm{erf}(\sqrt{\varepsilon}\cdot\alpha)} = 2\alpha^2\exp(\varepsilon\alpha^2), \quad (40)$$

$$u_0'(1) = \frac{2\sqrt{\varepsilon}\cdot\alpha\exp(-\varepsilon\alpha^2)}{\sqrt{\pi}\,\mathrm{erf}(\sqrt{\varepsilon}\cdot\alpha)} = 2\alpha^2. \quad (41)$$

To develop consistent asymptotic expansions, we require that $u_0'(1)$ of (41) matches with the first term of (29). This immediately implies that the Neumann relation of (8) must hold. So we see that the present solution admits Neumann's solution as the leading term of the asymptotic expansion as $t \to \infty$. Note that (8) gives much simpler expressions compared to the second terms of the RHS of (40) and (41), respectively. Now comparing the first term of (37) with that of (28) we have

$$d_1 = \alpha^2\exp(\varepsilon\alpha^2). \quad (42)$$

u_1 *and coefficients* C_2, d_2

u_1 satisfying (35) is

$$u_1 = -\frac{b_1}{2}(1-u_0) + \frac{Y}{2}u_0' - \frac{b_1}{2}\exp(-Y^2)$$
$$-\frac{\alpha^2 - b_1\exp(-\varepsilon\alpha^2)/2}{Dw(\sqrt{\varepsilon}\cdot\alpha)}Dw(Y), \quad (43)$$

where $b_1 = -2d_1/C_2$ from (32) and Dw denotes the

Dawson integral [10] defined as

$$Dw(Y) = \exp(-Y^2)\int_0^Y \exp(t^2)\,dt. \tag{44}$$

Hence the first derivative of u_1 is

$$u_1'(\eta) = \sqrt{\varepsilon}\cdot\alpha\left[\frac{1-b_1}{2}u_0' + \frac{Y}{2}u_0'' + b_1 Y \times \exp(-Y^2) \times \frac{\alpha^2 - b_1\exp(-\varepsilon\alpha^2)/2}{Dw(\sqrt{\varepsilon}\cdot\alpha)}\{1-2YDw(Y)\}\right]. \tag{45}$$

Accordingly

$$u_1'(0) = \left(1+\frac{2d_1}{C_2}\right)d_1 - \frac{\alpha^2(1+1/C_2)}{Dw(\sqrt{\varepsilon}\cdot\alpha)}, \tag{46}$$

$$u_1'(1) = \alpha^2\left[-1+2d_1/C_2 - 2/C_2 - \left(1+\frac{1}{C_2}\right) \times\left(\frac{1}{Dw(\sqrt{\varepsilon}\cdot\alpha)} - 2\sqrt{\varepsilon}\cdot\alpha\right)\right]. \tag{47}$$

First comparing with the second term of the LB expansion of g_2, namely (29) and (38), we have

$$C_2 = -1. \tag{48}$$

From the LB expansion for g_1, we have

$$d_2 = 0. \tag{49}$$

Equations (48) and (49) imply that u_1 of (43) simplifies considerably to

$$u_1 = -d_1(1-u_0) + \frac{Y}{2}u_0' - d_1\exp(-Y^2). \tag{50}$$

u_2 *and coefficients* C_{30}, d_{30}, d_{31} *and* C_{31}

u_2 satisfying (36) is

$$u_2 = \frac{b_2}{4}(1-u_0) + \frac{a_2-b_1}{2}u_1 + \frac{1}{2}(Yu_1' + 2u_1) - \frac{d_1}{2\sqrt{\varepsilon}\cdot\alpha} \times\left[1-2d_1+\frac{b_2}{2\alpha^2}Dw(\sqrt{\varepsilon}\cdot\alpha)\right]\exp(-Y^2) + \frac{b_2}{4}Dw_1(Y), \tag{51}$$

where

$$Dw_1(Y) = -1 + 2YDw(Y) \tag{52}$$

$$u_2'(\eta) = \sqrt{\varepsilon}\cdot\alpha\left[-\frac{b_2}{4}u_0' + \frac{a_2-b_1+3}{2}u_1' + \frac{Y}{2}u_1'' - \frac{d_1}{2\sqrt{\varepsilon}\cdot\alpha}\left\{1-2d_1+\frac{b_2}{2}Dw_1(\sqrt{\varepsilon}\cdot\alpha)\right\}\exp(-Y^2) \times(1-2Y^2) + \frac{b_2}{2}(1-2Y^2)Dw(Y) + \frac{b_2}{2}Y\right]. \tag{53}$$

Using (48) and (49), the constants given in (32) simplify to

$$a_2 = -2-4C_{30}, \quad b_2 = 2d_1(1-d_1-2C_{30}). \tag{54}$$

Hence we have

$$u_2'(0) = \frac{d_1}{2}\left[-2d_1^2 - \frac{d_1(1-d_1)}{2\alpha^2}Dw_1(\sqrt{\varepsilon}\cdot\alpha) + C_{30}\left\{12d_1 - 4 + \frac{2d_1}{\alpha^2}Dw_1(\sqrt{\varepsilon}\cdot\alpha)\right\}\right], \tag{55}$$

$$u_2'(1) = \frac{\alpha^2}{2}\left[-2d_1^2 + \frac{d_1(1-d_1)}{\alpha^2} - 2\varepsilon\alpha^2 + C_{30}\left(12d_1 - 4 - \frac{2d_1}{\alpha^2}\right)\right]. \tag{56}$$

The constant C_{30} can be determined by matching the third term of (29) and (56) giving

$$C_{30} = \left[d_1\left(3d_1 - 1 - \frac{1-d_1}{2\alpha^2}\right) + \varepsilon\alpha^2\right] \Big/ \left[2\left(d_1 - 1 - \frac{d_1}{2\alpha^2}\right)\right]. \tag{57}$$

Now that $O(\tau^2\ln\tau)$ does not appear in the LB expansions of (28), d_{30} must satisfy

$$d_{30} = -\frac{d_1C_{30}}{2}. \tag{58}$$

The term of $O(\tau^2)$ then requires that

$$d_{31} = C_{30}d_1^2\left\{1+\frac{Dw_1(\sqrt{\varepsilon}\cdot\alpha)}{2\alpha^2}\right\} + \frac{d_1^2}{2} \times\left\{1-3d_1 - \frac{1-d_1}{2\alpha^2}Dw_1(\sqrt{\varepsilon}\cdot\alpha)\right\} - \frac{d_1C_{31}}{2}. \tag{59}$$

Since C_{31} is still arbitrary, d_{31} remains indeterminate. Within the framework of the present asymptotic analysis, the constant C_{31} may not be determined. This reflects the fact that the initial condition at $t = 0$ may not be enforceable.

Putting $\bar{t} = \alpha^2 t$, the present asymptotic solutions are summarized below.

$$X \sim 2\alpha\sqrt{t} - 1 + \frac{C_{30}\ln\alpha^2 t}{\alpha\sqrt{t}} + \frac{C_{31}}{\alpha\sqrt{t}} + \ldots \tag{60}$$

$$u_w \sim \frac{\alpha\exp(\varepsilon\alpha^2)}{\sqrt{t}} + \frac{d_{30}\ln\alpha^2 t}{\alpha^3 t^{3/2}} + \frac{d_{31}}{\alpha^3 t^{3/2}} + \ldots \tag{61}$$

$$g_1 = X\frac{u_w}{1-u_w} = p_1(\tau) \sim 2d_1 + d_1(1-2d_1)\tau + (2d_1^3 - d_1^2 + 4C_{30}d_1^2 - 2C_{30}d_1 + C_{31}d_1 - 2d_{31})\tau^2 + \ldots \tag{62}$$

$$g_2 = \frac{X\dot{X}\alpha^2}{1-u_w} = p_2(\tau) \sim 2\alpha^2 + (1-2d_1)\alpha^2\tau + \alpha^2(2d_1^2 - d_1 + 4C_{30}d_1)\tau^2 + \ldots. \tag{63}$$

Here d_1, C_{30}, d_{30} and d_{31} are given in (42), (57)–(59) while C_{31} remains arbitrary.

6. INVERSION OF LB EXPANSIONS

In Stefan problems, we want to determine, among all variables, X and u_w functions. To determine them, one may either evaluate (60) and (61) asymptotically or invert the LB expansions of (22) and (23). The details of the inversion scheme are demonstrated below. It is instructive to point out here that, unlike the expansions of (60) and (61) which contain an arbitrary constant C_{31}, the coefficients in the interfacial LB expansions are fully specified. To show this fact specifically, we directly substitute the values of temperature derivatives obtained in Section 5 into (22) and (23) to obtain

$$p_1 \sim 2d_1 + d_1(1-2d_1)\tau + \frac{d_1}{2}\left[-2d_1^2 - \frac{d_1(1-d_1)}{\alpha^2} Dw_1(\sqrt{\varepsilon}\cdot\alpha) + 2C_{30} \times \left\{6d_1 - 2 + \frac{d_1}{\alpha^2} Dw_1(\sqrt{\varepsilon}\cdot\alpha)\right\}\right]\tau^2 + \ldots, \quad (64)$$

$$p_2 \sim 2\alpha^2 + (1-2d_1)\alpha^2\tau + \frac{\alpha^2}{2}\left[-2d_1^2 + \frac{d_1(1-d_1)}{\alpha^2} - 2\varepsilon\alpha^2 + 2C_{30}\left\{6d_1 - 2 - \frac{d_1}{\alpha^2}\right\}\right]\tau^2 + \ldots. \quad (65)$$

We should note that by the particular choice of constants d_1, C_{30}, C_{31}, d_{31} as found in Section 5, they obviously agree with the expansions of (62) and (63). It is easy to see that d_{31} is so chosen in (62) so as to cancel the indeterminate constant C_{31} from the LB expansions. This may suggest that X and u_w can be determined fully without encountering indeterminate constants thus contradicting the expansion of (60); but an arbitrary constant always appears at $O(t^{-1/2})$ in the inverted results as we shall show below. To invert, first truncate the LB expansions of (64) and (65) at the kth power of τ, eliminate u_w from p_1 and p_2 terms of (62) and (63), solve the polynomial equation for τ, thus setting up the first-order differential equation for $X^{(k)}$. This is carried out for $k = 0$–2 below.

$O(\tau^{(0)})$

First we truncate the LB expansions of (64) and (65) at $O(\tau^{(0)})$. Then eliminating $u_w^{(0)}$, one sets up a first-order differential equation for $X^{(0)}$ which gives when integrated

$$X^{(0)} = \sqrt{4\alpha^2 t + \{2\alpha^2 \exp(\varepsilon\alpha^2)\}^2 + C} - 2\alpha^2 \exp(\varepsilon\alpha^2), \quad (66)$$

$$u_w^{(0)} = \frac{2\alpha^2 \exp(\varepsilon\alpha^2)}{\sqrt{4\alpha^2 t + 4\alpha^2 \exp^2(\varepsilon\alpha^2) + C}}. \quad (67)$$

Here a superscript (k) denotes the solution from the LB expansions truncated at $O(\tau^{(k)})$. C is an integration constant and remains arbitrary within the framework of the present asymptotic analysis. $X^{(0)}$ and $u_w^{(0)}$ have the following asymptotic expansions.

$$X^{(0)} \sim 2\alpha\sqrt{t} - 2\alpha^2 \times \exp(\varepsilon\alpha^2) + \frac{C + 4\alpha^4 \exp^2(\varepsilon\alpha^2)}{4\alpha\sqrt{t}} + \ldots, \quad (68)$$

$$u_w^{(0)} \sim \frac{\alpha \exp(\varepsilon\alpha^2)}{\sqrt{t}} - \frac{\exp(\varepsilon\alpha^2)\{C + 4\alpha^4 \exp^2(\varepsilon\alpha^2)\}}{8\alpha t^{3/2}} + \ldots. \quad (69)$$

Note that the leading terms of both (60) and (61) are correctly reproduced as it should be. Arbitrariness remains at $O(t^{-1/2})$ and $O(t^{-3/2})$ for X and u_w, respectively, as in (60) and (61).

$O(\tau)$

Following the same procedure, $X^{(1)}$ and $u_w^{(1)}$ can be again obtained in closed form as

$$X^{(1)} + 2(d_1+1)\sqrt{[X^{(1)}]^2 + 4(1-d_1)X_1 + 4d_1^2} + 4(1-d_1)\ln\{2X^{(1)} + 2(1-d_1) + 2\sqrt{[X^{(1)}]^2 + 4(1-d_1)X^{(1)} + 4d_1^2}\} + [X^{(1)}]^2 + 4d_1 X^{(1)} = 8\alpha^2 t + C \quad (70)$$

$$u_w^{(1)} = \frac{2d_1 + d_1(1-2d_1)[X^{(1)}\dot{X}^{(1)} - 2\alpha^2]}{2d_1 + d_1(1-2d_1)[X^{(1)}\dot{X}^{(1)} - 2\alpha^2] + X^{(1)}}. \quad (71)$$

For $t \to \infty$,

$$X^{(1)} \sim 2\alpha\sqrt{t} - 1 - \frac{1-2d_1}{3\alpha\sqrt{t}}\ln\alpha^2 t + \frac{1 - d_1 - d_1^3 + C}{2\alpha\sqrt{t}} + \ldots,$$

$$u_w^{(1)} \sim \frac{\alpha\exp(\varepsilon\alpha^2)}{\sqrt{t}} + \frac{d_1(1-2d_1)}{\alpha^3 t^{3/2}}\ln\alpha^2 t + \frac{d_1}{2\alpha^3 t^{3/2}}\{d_1(1-2d_1) - C\} + \ldots. \quad (72)$$

$X^{(1)}$ and $u_w^{(1)}$ again reproduce the asymptotic expansions (60) and (61) correct to the second terms of the expansions. It is most interesting to note that the present two-term expansions reproduce logarithmic terms which appear only in the next order of the classical asymptotic expansions of (60) and (61).

$O(\tau^{(2)})$

To carry out the inversion, a numerical approach is far simpler from this order on. For example, elimination of u_w from (64) and (65) gives a cubic equation in τ at this order. This is easily solved by Newton's method giving

$$h(t) = \frac{X^{(2)}}{\alpha^2}\frac{dX^{(2)}}{dt} - 2 = F[X^{(2)}; d_1, C_{30}, \varepsilon, \alpha]. \quad (73)$$

The solution F is a function of $X^{(2)}$ with additional dependence on parameters such as d_1, etc. Equation (73) can be solved numerically by a Runge–Kutta method starting at sufficiently large values of $X = 20$. The numerical results are tabulated in Table 1 and compared with the finite-difference solution of the problem [11]. Initial values of t at $X = 20$ are estimated from (60) retaining the appropriate order of the

Table 1. Constants $C_{30}, C_{31}, d_{30}, d_{31}$

ε	C_{30}	C_{31}	d_{30}	d_{31}
0.0	0.0	0.250	0.0	−0.0625
0.1	−0.0278	0.34	0.0071	−0.095
0.5	−0.1174	0.60	0.0315	−0.19
1.0	−0.1953	0.90	0.0551	−0.36
2.0	−0.2607	1.4	0.0770	−0.60

approximations. For example, to compute $t^{(2)}$ we need C_{31}. We discuss in Section 7 how these constants are estimated from the numerical solution of [11]. Then we see that the inverted results for X and u_w give quite good agreement with the numerical solution even at such small values of X as 2.

7. DISCUSSIONS

We have obtained the first three terms of the asymptotic expansions for X and u_w functions consistent with the LB expansions of the convection Stefan problem with surface radiation valid for large values of time. We have seen that a logarithmic term appears at the third term of $(\ln t/\sqrt{t})$ with an undetermined constant C_{31} appearing at $O(1/\sqrt{t})$. The constant can only be determined if the initial condition of (26) can be taken into account. Before estimating the constant, we will show below that our results reproduce the exact solution of (11) and (12) with $\varepsilon = 0$ correctly. Now if $\varepsilon = 0$, (8) gives

$$d_1 = \alpha^2 = 1/2, \quad C_{30} = d_{30} = 0. \tag{74}$$

In fact, for $\varepsilon \ll 1$, we have

$$C_{30} \sim -\frac{7}{24}\varepsilon \quad \text{and} \quad d_{30} \sim -\frac{7}{96}\varepsilon.$$

Now the large time expansion of (14) gives

$$C_{31} = 1/4. \tag{75}$$

Then (59) with $\varepsilon = 0$ gives

$$d_{31} = -1/16. \tag{76}$$

Equation (76) is verified by the second term of the large time expansion of u_w of (14). So our results reproduce the exact solution of (11) and (12) asymptotically correct to this order, if $C_{31} = 1/4$. Also we note that (74) implies that logarithmic terms disappear altogether at $\varepsilon = 0$ and the solution is regular everywhere provided a cut is inserted between $-1/2$ and $-\infty$ [see equation (11)].

Now for nonzero ε's we may estimate C_{31} from the finite-difference solution of the problem. Using (60), C_{31} may be estimated, with reasonable accuracy, by (77) below if X is sufficiently large.

$$C_{31} \sim \alpha\sqrt{t}\left[X - \left\{2\alpha\sqrt{t} - 1 + \frac{C_{30}\ln \alpha^2 t}{\alpha\sqrt{t}}\right\}\right]. \tag{77}$$

Table 2. Numerical results of interface position

Position	Inverted LB expansion			Finite-difference	Asymptotic expansion equation (60)
X	$t^{(0)}$	$t^{(1)}$	$t^{(2)}$	[11]	$t^{(2)}$
		$\varepsilon = 0.1$			
1.0	1.57	2.02	1.52	1.53	1.22
2.0	4.67	4.60	4.12	4.10	3.95
5.0	18.16	18.56	18.10	17.97	18.01
10.0	62.14	62.47	62.0	—	61.98
20.0	227.6	227.7	227.3	—	227.3
		$\varepsilon = 0.5$			
1.0	1.82	2.06	1.71	1.65	—
2.0	4.80	4.98	4.16	4.50	3.86
5.0	20.67	20.67	20.41	20.12	20.02
10.0	70.27	69.90	69.70	—	69.50
20.0	256.2	255.1	255.3	—	255.3
		$\varepsilon = 1.0$			
1.0	2.18	2.09	1.82	1.80	—
2.0	5.54	5.41	5.13	4.99	3.24
5.0	18.45	23.10	22.90	22.49	22.13
10.0	79.71	78.46	78.43	—	78.06
20.0	289.0	286.6	286.8	—	286.8
		$\varepsilon = 2.0$			
1.0	2.67	2.29	1.49	2.05	—
2.0	6.93	6.38	5.47	5.83	8.72
5.0	29.23	28.02	27.18	26.78	25.80
10.0	98.17	95.7	95.05	—	94.45
20.0	352.2	350.0	349.6	—	349.6

When estimated by (77), for example, C_{31} of 1/4 of (75) is computed to be 0.247 and 0.248, respectively, where the exact values of ($X = 4, t = 12$) and ($X = 5, t = 17.5$) are used. So we believe (77) gives an adequate accuracy for practical purpose. C_{31}'s are computed for several values of ε by (77) using the numerical values of [11] at $X = 4$ and $X = 5$. Results are tabulated in Table 1 together with other constants such as d_{30}, C_{30}, etc. Once these constants are estimated, (60) can be solved for t by Newton's method. The results are tabulated in Table 2 for comparison with the inverted results of the LB expansions of Section 6. The inverted results seem to agree better with the numerical results even at small values of X.

It is interesting to note that although the expansions for X and u_w suffer a logarithmic term, the LB expansions do not contain any logarithmic term at least up to the order considered. It may appear at higher orders but it is possible that singularities in X and u_w are such that they cancel each other when combined to form g_1 and g_2 functions. Within the framework of the present analysis, it is impossible to make a definite conclusion on this point.

Since the initial conditions at $t = 0$ are irrelevant, the present solution can be used to describe a large time behavior of a wide class of Stefan problems provided the problem settles down to the Neumann solution.

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UNE SOLUTION ASYMPTOTIQUE, POUR UN TEMPS GRAND, DU PROBLEME DE CONVECTION DE STEFAN, AVEC RAYONNEMENT DE SURFACE

Résumé—Une solution asymptotique du problème de convection de Stefan pour un temps grand est obtenue dans le cas d'un rayonnement de surface. Le problème de frontière mobile est reformulé en un problème de frontière fixe où les développements de Lagrange–Bürmann sont utilisés pour compléter la transformation. Une solution asymptotique du problème est obtenue sous réserve que les développements asymptotiques admis pour la position $X(t)$ de l'interface et la température de paroi $u_w(t)$ pour des temps grands sont compatibles avec les développements de Lagrange–Bürmann à l'interface. On trouve que les développements asymptotiques admettent la solution de Neumann comme terme principal et que les termes logarithmiques commencent à intervenir dans les termes de troisième ordre des développements pour un nombre de Stefan nul.

EINE ASYMPTOTISCHE LANGZEITLÖSUNG DES KONVEKTIVEN STEFANPROBLEMS MIT OBERFLÄCHENSTRAHLUNG

Zusammenfassung—Eine asymptotische Langzeitlösung wurde für das konvektive Stefanproblem mit Oberflächenstrahlung ermittelt. Das Problem beweglicher Begrenzungen wurde in ein Problem mit festen Grenzen umformuliert. Bei der Variablen-Transformation wurden Lagrange–Bürmann-Entwicklungen angewandt. Eine asymptotische Lösung des Problems erhält man mit der Annahme, daß die asymptotischen Entwicklungen für die Grenzflächenposition $X(t)$ und die Wandtemperatur $u_w(t)$ für lange Zeiten mit den nach Lagrange–Bürmann ermittelten übereinstimmen. Die asymptotischen Erweiterungen beeinhalten als Hauptterme die Neumann'sche Lösung. Logarithmische Ausdrücke beginnen bei Termen dritter Ordnung der Entwicklungen einzugreifen, wenn die Stefanzahl von Null verschieden ist.

АСИМПТОТИЧЕСКОЕ РЕШЕНИЕ НА БОЛЬШИХ ВРЕМЕНАХ КОНВЕКТИВНОЙ ЗАДАЧИ СТЕФАНА С УЧЕТОМ ИЗЛУЧЕНИЯ ПОВЕРХНОСТИ

Аннотация—Для конвективной задачи Стефана, включающей излучение поверхности, получено асимптотическое решение, справедливое на больших временах. С помощью преобразования Лагранжа–Бюрмана совершен переход к задаче с неподвижными границами. Исходя из условия, что предполагаемые асимптотические разложения для положения границы раздела $X(t)$ и температуры стенки при больших временах согласуются с разложениями Лагранжа–Бюрмана для границы раздела получено асимптотическое решение. Найдено, что решение Неймана представляет собой главный член асимптотического разложения и что в членах третьего порядка малости появляются логарифмические выражения.